The stability of plane Poiseuille flow of a dusty gas

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In this paper some approximate results are presented for the problem of the stability of plane Poiseuille flow of a dusty gas, following a formulation given recently by Saffman (1962). It is assumed that the mass concentration of the dust, f, is small, and results are obtained by making a perturbation of the curve of neutral stability for a clean gas, using the approximate solutions given by Stuart (1954). The perturbation equation is expressed in terms of integrals by introducing the adjoint wave function, the calculation of which is described. The integral coefficients were evaluated by numerical integration using a Mercury computer, and the results are illustrated for $f = 0{\cdot}05$ by a set of perturbed neutral stability curves at different values of the time relaxation parameter SR varying from 0 to 500. These results, whilst not of great numerical accuracy, are sufficient to show qualitatively how the curve of neutral stability is modified by the presence of the dust.

1. Introduction

In a recent paper Saffman (1962) has given a formulation of the problem of the linearized stability of a plane parallel flow of a dusty gas, in which the dust is represented macroscopically in terms of a number density of very small particles. When the dust particles move relative to the gas it is assumed that Stokes's law of resistance applies, in which case the resulting mathematical problem is as follows.

Let the gas, which is assumed incompressible, have undisturbed velocity $U(y)$ in the x direction. The effect of sedimentation is neglected and it is assumed that in the undisturbed state the dust is carried along with the gas, i.e. with the same velocity $U(y)$. With a plane perturbation stream function of the usual form $\phi(y)\exp\{i\alpha(x-ct)\}$, Saffman deduced the Orr–Sommerfeld equation for the dusty gas, which can be written in dimensionless form

$$(D^2-\alpha^2)^2\phi = i\alpha R\{(\overline{u}-c)(D^2-\alpha^2)\phi-(D^2\overline{u})\phi\}, \tag{1}$$

where
$$\overline{u} = U+\frac{f(U-c)}{1+i\alpha(U-c)SR},$$

$D \equiv d/dy$, α is a real wave-number, R the Reynolds number, $\overline{u}$ is a modified (complex) velocity profile involving the mass concentration of the dust f, $SR = U\tau/L$, and τ represents the scale of the time which dust particles take to follow changes in the gas velocity. A difficult characteristic-value problem is posed by this equation together with no-slip boundary conditions on two walls $y = \pm 1$, at which $\phi = D\phi = 0$.

Saffman drew attention to two limiting cases in which the effect of the dust can easily be seen. When SR, or τ, is small the dust follows the velocity fluctuations of the gas without time delay, and the result is to increase the effective density of the gas without changing its viscosity. This reduces the value of R for neutral stability in the ratio $1/(1+f)$, as may be seen by approximating $\overline{u} \sim U+f(U-c)$ in (1). When SR is very large the dust is unaffected by the fluctuations in the gas velocity, and the flow is stabilized by the dust damping the perturbation in the gas flow. This is demonstrated mathematically by the approximation $\overline{u} \sim U+(f/i\alpha SR)$ at large SR, which implies that the problem is equivalent to that of a clean gas with a positive time amplification factor f/SR.

The author has taken up some aspects of the mathematical problem in more detail, and this paper will describe the results obtained. In particular, the case of plane Poiseuille flow for which $U = 1-y^2$ is studied. Following the classical problem, we consider a disturbance wave function $\phi(y)$ which is an even function of y, so that we need only apply the two boundary conditions $\phi = D\phi = 0$ at $y = +1$. The problem is made considerably more difficult by the presence of two extra independent parameters f and S (or τ) in the equation, and we have not so far attempted a formal solution of the problem. Rather, taking account of the fact that the formulation of the problem will be valid only for small values of f, we have performed a linear perturbation of the neutral stability curve for the clean gas $f = 0$, which will give the results likely to be of practical value.

2. Perturbation problem for small f

Let (α_0, R_0) be a point on the curve of neutral stability in the (α, R) plane for the clean gas, with the corresponding wave function $\phi_0(y)$ and real wave velocity $c = c_0$. We write perturbed quantities to the first power in f,

$$\alpha = \alpha_0+\lambda f, \quad R = R_0+\mu f, \quad c = c_0+\nu f, \quad \phi = \phi_0+f\chi(y). \tag{2}$$

In general ν may be a complex coefficient, but we shall restrict our calculation to finding the perturbed neutral stability curve only, in which case ν will be real.

When we substitute from (2) into (1), cancel out the terms of order f^0, and put the coefficient of f^1 equal to zero, we obtain an equation for the perturbation stream function $\chi(y)$,

$$\begin{aligned}(D^2-\alpha^2)^2\chi - i\alpha R\{(U-c)(D^2-\alpha^2)\chi-(D^2U)\chi\} \\ = 4\alpha\lambda(D^2-\alpha^2)\phi+i(\lambda R+\alpha\mu)\{(U-c)(D^2-\alpha^2)\phi-(D^2U)\phi\} \\ +i\alpha R\left\{\left[\frac{U-c}{1+i\alpha SR(U-c)}-\nu\right](D^2-\alpha^2)\phi-2\alpha\lambda(U-c)\phi-\phi D^2\left[\frac{U-c}{1+i\alpha SR(U-c)}\right]\right\},\end{aligned} \tag{3}$$

where the suffix 0 in α, R, c and ϕ has now been dropped. Equation (3) is an inhomogeneous Orr–Sommerfeld equation. Except for the coefficients λ, μ, ν, the functions occurring on the right-hand side may be regarded as known from the data at the unperturbed point on the clean gas curve. Formally we require to solve equation (3) for $\chi(y)$ subject to the conditions $\chi = D(\chi) = 0$ at $y = 1$, it being supposed that $\chi(y)$ like $\phi(y)$ is an even function of y. The solution for $\chi(y)$ will be of the form

$$\chi = A_1\chi_1 + A_2\chi_2 + \lambda\chi_\lambda + \mu\chi_\mu + \nu\chi_\nu + \tilde{\chi},$$

in which the first two terms give the two independent even solutions of the homogeneous equation with the right-hand side zero, and the remaining terms are particular integrals grouped according to their coefficient λ, μ or ν, with $\tilde{\chi}$ representing the term without any of these coefficients. Substitution in the boundary conditions gives two equations of the form

$$\sigma_{11}A_1 + \sigma_{12}A_2 = k_1 + l_1\lambda + m_1\mu + n_1\nu,$$

$$\sigma_{21}A_1 + \sigma_{22}A_2 = k_2 + l_2\lambda + m_2\mu + n_2\nu.$$

Since a linear combination of χ_1 and χ_2 is an eigenfunction at the unperturbed point,

$$\begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{vmatrix} = 0,$$

and it is then necessary that

$$\begin{vmatrix} \sigma_{11} & k_1 + l_1\lambda + m_1\mu + n_1\nu \\ \sigma_{21} & k_2 + l_2\lambda + m_2\mu + n_2\nu \end{vmatrix} = 0. \qquad (4)$$

The result is therefore one linear relation between λ, μ, ν, with in general complex coefficients, which can be separated into two real simultaneous linear equations. The values of λ, μ, ν are thus not uniquely determined by this procedure, because the direction in which the perturbation from one curve to the other is made needs to be specified. We may, for example, choose perturbations at (i) constant α, (ii) constant R, or (iii) constant c, for which $\lambda = 0$, $\mu = 0$, or $\nu = 0$, respectively. So far as the representation in the (α, R)-plane is concerned we could imagine ν eliminated between the two equations, and a single linear relation between λ and μ then follows, which gives a straight line of perturbation points for the new curve. If one is to follow through this method the central problem is to find suitable expressions for the particular integrals $\tilde{\chi}$, χ_λ, χ_μ and χ_ν, bearing in mind that these will still contain one of the extra parameters τ. One can find without difficulty series in powers of SR or inverse powers of SR, which will be suitable at small or large values of SR respectively. However, it is difficult to fill in the cases where $SR \sim 1$ by this method, and to overcome this we have resorted to another method suggested by Dr J. T. Stuart which makes use of the adjoint wave function.

In recent papers by Stuart (1960) and Watson (1960, 1962), the inhomogeneous Orr–Sommerfeld equation arises, and they have shown how the adjoint wave function may be used to derive integral properties of the solutions. We now discuss briefly the adjoint wave function and the way in which it may be used in the present problem.

3. Properties of the adjoint function

In our case the differential form of interest is

$$L(\phi) \equiv (D^2-\alpha^2)^2\phi - i\alpha R[(U-c)(D^2-\alpha^2)\phi-(D^2U)\phi].$$

The adjoint form, which we denote by $\tilde{L}(\tilde{\phi})$, follows easily from the definition (see Ince 1944, §9.31), and is

$$\tilde{L}(\tilde{\phi}) \equiv (D^2-\alpha^2)^2\tilde{\phi} - i\alpha R[(U-c)(D^2-\alpha^2)\tilde{\phi}+2(DU)D\tilde{\phi}].$$

It follows that
$$\int_a^b \{\tilde{\phi}L(\phi)-\phi\tilde{L}(\tilde{\phi})\}\,dy = [P(\phi,\tilde{\phi})]_a^b,$$

where
$$\begin{aligned}P(\phi,\tilde{\phi}) = \phi[D\{(2\alpha^2+i\alpha R(U-c))\tilde{\phi}\}-D^3\tilde{\phi}]\\ -(D\phi)[\{2\alpha^2+i\alpha R(U-c)\}\tilde{\phi}-D^2\tilde{\phi}]\\ -(D^2\phi)(D\tilde{\phi})+(D^3\phi)\tilde{\phi}.\end{aligned} \tag{5}$$

In particular, if $\phi(y)$ is the wave function for the clean gas problem $L(\phi)=0$, and if $\tilde{\phi}$ is a solution of the adjoint equation $\tilde{L}(\tilde{\phi}\}=0$, we have with $a=0$, $b=1$,

$$[P(\phi,\tilde{\phi})]_0^1 = 0. \tag{6}$$

In our problem, in which $U=1-y^2$, ϕ will be an even function of y which satisfies the conditions $\phi=D\phi=0$ at $y=1$, and, if we similarly restrict $\tilde{\phi}$ to be an even solution of $\tilde{L}(\tilde{\phi})=0$, we then have from (5) and (6)

$$(D^2\phi)(D\tilde{\phi}) = \tilde{\phi}(D^3\phi) \quad \text{at} \quad y=1. \tag{7}$$

It was shown by Stuart (1960) that, in this particular case, if $\psi(y)$ satisfies the equation $L(\psi)=0$, then $\tilde{\psi}=(D^2-\alpha^2)\psi$ is a solution of the equation $\tilde{L}(\tilde{\psi})=0$. It follows that if ψ_1, ψ_2 are the two independent even solutions of $L(\psi)=0$, then $\tilde{\psi}_1=(D^2-\alpha^2)\psi_1$, $\tilde{\psi}_2=(D^2-\alpha^2)\psi_2$ are two independent even solutions of $\tilde{L}(\tilde{\psi})=0$. Hence we may write $\tilde{\phi}$ as any linear combination of $\tilde{\psi}_1$, $\tilde{\psi}_2$, i.e. $\tilde{\phi}=\tilde{\psi}_1+\tilde{\delta}\tilde{\psi}_2$. We can satisfy one boundary condition formally on $\tilde{\phi}$ by making use of $\tilde{\delta}$. It is most convenient to use $\tilde{\delta}$ to satisfy the condition $\tilde{\phi}=0$ at $y=1$. It then follows from (7) that $D\tilde{\phi}=0$ at $y=1$. (We cannot here impose two arbitrary homogeneous boundary conditions on $\tilde{\phi}$ because the parameter c has already been fixed.)

Coming now to the application to the inhomogeneous Orr–Sommerfeld equation, suppose that

$$L(\phi) = k(y), \tag{8}$$

where $k(y)$ is an even function of y, and ϕ is an even function of y satisfying the conditions $\phi=D\phi=0$ at $y=1$. Then using the adjoint function $\tilde{\phi}$ as defined above, and substituting in (5) with $a=0$, $b=1$, we have

$$\int_0^1 \tilde{\phi}(y)\,k(y)\,dy = 0. \tag{9}$$

In this context we identify equation (8) with equation (3), writing χ instead of ϕ. Hence it follows from (9) that

$$4\alpha\lambda\int_0^1 \tilde{\phi}(D^2-\alpha^2)\,\phi\,dy+i(\lambda R+\alpha\mu)\int_0^1 \tilde{\phi}\{(U-c)\,(D^2-\alpha^2)\,\phi-(D^2U)\,\phi\}\,dy$$
$$+i\alpha R\int_0^1 \tilde{\phi}\left[\left\{\frac{U-c}{1+i\alpha SR(U-c)}-\nu\right\}(D^2-\alpha^2)\,\phi-2\alpha\lambda(U-c)\,\phi\right.$$
$$\left.-\phi D^2\left\{\frac{U-c}{1+i\alpha SR(U-c)}\right\}\right]dy=0. \qquad (10)$$

Equation (10) is in effect an alternative form of the result (4), but has the advantage of giving an explicit form for the coefficients in the simultaneous equations. Looked at from this point of view the solution is in two parts: (i) the calculation of the adjoint wave function $\tilde{\phi}$ and (ii) the evaluation of the integrals arising in (10). We now describe the steps taken to obtain a first approximation for the adjoint function.

4. Calculation of $\tilde{\phi}$

The profile of the adjoint function may be calculated in general at any point in the (α, R)-plane, when the corresponding value of c is known from a solution of the characteristic-value problem for ϕ. The main interest at present is in the

	A	B	C	D	E
α	1·02	0·96	0·80	0·68	0·58
R	5,460	6,320	10,350	19,400	44,600
	F	G	H	I	J
α	0·86	1·00	1·10	1·17	1·18
R	140,000	49,600	21,200	10,150	7,050

TABLE 1. Values of α and R for neutral stability of plane Poiseuille flow, after Stuart (1954).

points on the neutral curve where c is real. Our calculations provide approximate values of the real and imaginary parts of $\tilde{\phi}$ at ten points of the neutral curve given approximately by Stuart (1954), which are labelled A, B, ..., J, according to table 1.

Following the notation of Stuart we denote the two independent even solutions of $L(\psi) = 0$ by v_1 and v_3. These solutions have special features near the transition from stable to unstable flow because this transition takes place at large values of R. The function v_1 is well approximated by an even solution of the inviscid equation obtained by neglecting the viscous terms in $L(\psi) = 0$, i.e.

$$(U-c)\,(D^2-\alpha^2)\,\psi-(D^2U)\,\psi=0. \qquad (11)$$

This approximate solution becomes inadequate in the neighbourhood of the critical layer where $U = c$, when c is real. At this point one solution of (11) is singular and the viscous terms then become important in determining the transition of the solution across the layer. Thus at points away from the critical layer, where v_1 is effectively a solution of (11), $\tilde{v}_1 = (D^2-\alpha^2)\,v_1$ is given approximately by

$$\tilde{v}_1 \simeq \frac{(D^2U)}{(U-c)}v_1 = -\frac{2v_1}{(y_0^2-y^2)}, \tag{12}$$

where y_0 denotes the level of the critical layer.

To calculate corrections to this inviscid approximation to $\tilde{v}_1$, we have to take account of the solutions of the full equation in the critical layer in which the variable η is used, where $y-y_0 = \epsilon\eta$, $\epsilon = (\alpha R U_0')^{-\frac{1}{3}}$, and $U_0' = -2y_0$. A difficulty in calculating in terms of η is that the scaling of y varies from point to point. Bearing in mind that $\tilde{\phi}$ is intended to be used in numerical integrations using a computer, on which it is desirable to use a standard programme in each case, the calculation has been made for standard values of y given by $y = 0\ (0{\cdot}05)\ 0{\cdot}80\ (0{\cdot}02)\ 1{\cdot}0$. The smaller interval 0·02 was used between 0·8 and 1·0 to get a better description of the functions across the critical layer, where the variations are in some cases very erratic.

The following is a résumé of the stages of calculation, using a desk machine, for each of the points A, ..., J, in the above range of y for each case.

(i) *Inviscid approximation to* v_1

The following function given by Stuart (1954) was tabulated:

$$v_1 \sim (y_0^2-y^2)\,(1+\tfrac{1}{10}\alpha^2y^2)+\tfrac{4}{15}\alpha^2y_0^2\left\{y^2+\tfrac{1}{2}(y_0^2-y^2)\log\left(\frac{y_0^2}{y_0^2-y^2}\right)\right\} \quad (y < y_0),$$

$$v_1 \sim (y_0^2-y^2)\,(1+\tfrac{1}{10}\alpha^2y^2)+\tfrac{4}{15}\alpha^2y_0^2\left\{y^2+\tfrac{1}{2}(y_0^2-y^2)\left[\log\left(\frac{y_0^2}{y^2-y_0^2}\right)+i\pi\right]\right\} \quad (y > y_0).$$

(ii) *Viscous correction to* v_1

To follow the solution v_1 through the critical layer in more detail we rewrite the singular term as

$$\begin{aligned}\tfrac{1}{2}(y_0^2-y^2)\log\left(\frac{y_0^2}{y_0^2-y^2}\right) &= \tfrac{1}{2}(y_0+y)\,(-\epsilon\eta)\\ &\quad\times\{\log y_0^2-\log(y_0+y)-\log(-\epsilon)\}+\tfrac{1}{2}(y_0+y)\,\epsilon\eta\log\eta.\end{aligned}$$

Following the work of Tollmien (1929), the singularity $\eta\log\eta$ occurring in the last term is replaced by the function $\bar{S}(\eta)$ which is a solution of the equation

$$i\frac{d^4\bar{S}}{d\eta^4}+\eta\frac{d^2\bar{S}}{d\eta^2} = 1.$$

This function has been tabulated by Meksyn & Stuart (1951), and Holstein (1950), and the present calculations draw on the values given by Holstein. Replacing $\eta\log\eta$ by $\bar{S}(\eta)$ we find the viscous correction to v_1 as

$$\tfrac{2}{15}\alpha^2y_0^2\,[(y_0^2-y^2)\log\eta+\epsilon(y_0+y)\,\bar{S}(\eta)] \quad (\eta > 0),$$

and $$\tfrac{2}{15}\alpha^2y_0^2\,[(y_0^2-y^2)\log(-\eta)+\epsilon(y_0+y)\,\bar{S}(\eta)-\tfrac{1}{2}i\pi(y_0^2-y^2)] \quad (\eta < 0).$$

Combining these with the calculation (i), we obtain corrected values of v_1.

(iii) *Viscous integral* v_3

This function is given by

$$v_3 = \int_\infty^\eta d\eta \int_\infty^\eta \eta^{\frac{1}{2}} H_{\frac{1}{3}}^{(1)} [\tfrac{2}{3}(i\eta)^{\frac{3}{2}}]\, d\eta.$$

The required values have been interpolated from the tabulation of the function given by Holstein.

(iv) *The function* $\tilde{v}_1$

The inviscid approximation to $\tilde{v}_1$ given by (12) was first tabulated, but it was ultimately found that this is a very poor approximation to the function. A more accurate expression for $\tilde{v}_1$ was afterwards used, derived by a formal differentiation of v_1 as follows. From

$$v_1 = (y_0^2 - y^2)(1 + \tfrac{1}{10}\alpha^2 y^2) + \tfrac{4}{15}\alpha^2 y_0^2 \times \left[y^2 + \tfrac{1}{2}(y_0^2 - y^2)\log\left(\frac{y^2}{-\epsilon(y_0+y)}\right) + \tfrac{1}{2}\epsilon(y_0+y)\,\bar{S}(\eta)\right],$$

we deduce

$$D^2 v_1 = -2 - \tfrac{6}{5}\alpha^2 y^2 + \tfrac{4}{15}\alpha^2 y_0^2 \left[\frac{17}{4} - \frac{y_0}{y_0+y} - \log\left\{\frac{y_0^2}{-\epsilon(y_0+y)}\right\} + \frac{y_0+y}{2\epsilon}\bar{S}'' + \bar{S}'\right],$$

in which $\bar{S}'$, $\bar{S}''$ denote $d\bar{S}/d\eta$ and $d^2\bar{S}/d\eta^2$, respectively. These functions are also tabulated by Holstein, and one can then tabulate $D^2 v_1$ directly, and thereafter obtain $\tilde{v}_1 = (D^2 - \alpha^2)\, v_1$.

(v) *The function* $\tilde{v}_3$

This function was tabulated from the formula $\tilde{v}_3 = (D^2 - \alpha^2)\, v_3$, drawing on the values given by Holstein for $d^2 v_3/d\eta^2$.

(vi) *The adjoint function* $\tilde{\phi} = \tilde{v}_1 + \tilde{\delta}\tilde{v}_3$

$\tilde{\phi}$ was finally computed as a linear combination of $\tilde{v}_1$ and $\tilde{v}_3$ that makes $\tilde{\phi} = 0$ at $y = 1$. A check on the calculation is that $D\tilde{\phi}$ should be 0 at $y = 1$, which is generally confirmed by the trend of the first differences as $y \to 1$.

In table 2 we give the values of $\tilde{\phi}$ rounded off to three significant figures or decimal places. Figures 1–4 give the corresponding graphs.

We conclude this section by drawing attention to one point of significance in the classical problem which emerges from the calculation, namely that the inviscid approximation to v_1 is a much more accurate approximation than the inviscid approximation to $\tilde{v}_1$. This means that the critical layer is much more sharply defined for ϕ than it is for $\tilde{\phi}$. In illustration we give in figures 5 and 6 a plot of the real and imaginary parts of $\tilde{v}_1$ for case A, together with the inviscid approximation which has a singularity at $y = y_0$. The wide spread of the effect of viscosity on these functions is evident from the comparison of the curves. Further, it is to be noticed that $(D^2 - \alpha^2)\,\phi$ is a measure of the disturbance vorticity in the characteristic value problem for ϕ. This is a linear combination of $\tilde{v}_1$ and $\tilde{v}_3$. Hence the disturbance vorticity is spread by the action of viscosity well outside the critical layer associated with the velocity disturbance.

y	A $R(\tilde{\phi})$	A $I(\tilde{\phi})$	B $R(\tilde{\phi})$	B $I(\tilde{\phi})$	C $R(\tilde{\phi})$	C $I(\tilde{\phi})$	D $R(\tilde{\phi})$	D $I(\tilde{\phi})$	E $R(\tilde{\phi})$	E $I(\tilde{\phi})$
1·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0
0·98	0·157	−0·197	0·100	−0·271	0·065	−0·362	0·052	−0·43	0·085	−0·706
0·96	0·336	−0·824	0·323	−0·956	0·144	−1·19	0·003	−1·47	−0·221	−2·08
0·94	0·606	−1·81	0·393	−1·97	−0·068	−2·25	−0·570	−2·60	−1·45	−3·15
0·92	0·419	−2·95	0·018	−3·07	−0·763	−3·22	−1·70	−3·31	−3·10	−3·11
0·90	−0·239	−3·91	−0·689	−3·97	−1·88	−3·72	−3·04	−3·25	−4·18	−2·02
0·88	−1·35	−4·82	−2·03	−4·42	−3·12	−3·67	−4·05	−2·47	−4·19	−0·729
0·86	−2·77	−5·06	−3·17	−4·30	−4.13	−2·98	−4·42	−1·36	−3·61	0·033
0·84	−4·09	−4·54	−4·28	−3·64	−4·63	−1·97	−4·15	−0·426	−3·03	0·166
0·82	−5·06	−3·55	−4·98	−2·62	−4·60	−0·983	−3·61	0·053	−2·71	0·086
0·80	−5·48	−2·39	−5·15	−1·52	−4·20	−0·267	−3·12	0·176	−2·59	0·019
0·75	−4·68	−0·171	−4·06	0·083	−3·06	0·183	−2·61	0·025	−2·43	0·0026
0·70	−3·41	0·23	−3·07	0·164	−2·67	0·0244	−2·47	0·0015	−2·30	0·0
0·65	−2·88	0·058	−2·70	0·019	−2·50	0·0014	−2·36	0·0015	−2·23	0·0
0·60	−2·60	0·0	−2·50	0·0016	−2·38	0·002	−2·25	0·0	−2·18	0·0
0·55	−2·46	0·0	−2·40	0·0025	−2·30	0·0	−2·19	0·0	−2·14	0·0
0·50	−2·35	0·0	−2·35	0·0	−2·21	0·0	−2·15	0·0	−2·11	0·0
0·45	−2·26	0·0	−2·25	0·0	−2·16	0·0	−2·11	0·0	−2·08	0·0
0·40	−2·20	0·0	−2·17	0·0	−2·12	0·0	−2·09	0·0	−2·06	0·0
0·35	−2·15	0·0	−2·12	0·0	−2·09	0·0	−2·06	0·0	−2·05	0·0
0·30	−2·10	0·0	−2·09	0·0	−2·06	0·0	−2·04	0·0	−2·03	0·0
0·25	−2·07	0·0	−2·06	0·0	−2·04	0·0	−2·03	0·0	−2·02	0·0
0·20	−2·04	0·0	−2·04	0·0	−2·03	0·0	−2·02	0·0	−2·01	0·0
0·15	−2·02	0·0	−2·02	0·0	−2·01	0·0	−2·01	0·0	−2·007	0·0
0·10	−2·01	0·0	−2·01	0·0	−2·006	0·0	−2·00	0·0	−2·00	0·0
0·05	−2·00	0·0	−2·00	0·0	−2·00	0·0	−2·00	0·0	−2·00	0·0
0·00	−2·00	0·0	−2·00	0·0	−2·00	0·0	−2·00	0·0	−2·00	0·0

y	F $R(\tilde{\phi})$	F $I(\tilde{\phi})$	G $R(\tilde{\phi})$	G $I(\tilde{\phi})$	H $R(\tilde{\phi})$	H $I(\tilde{\phi})$	I $R(\tilde{\phi})$	I $I(\tilde{\phi})$	J $R(\tilde{\phi})$	J $I(\tilde{\phi})$
1·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0
0·98	1·64	0·178	1·08	0·087	0·676	0·0169	0·411	0·0027	0·351	−0·118
0·96	5·19	−3·37	3·63	−1·5	2·13	−0·898	1·45	−0·688	1·08	−0·708
0·94	2·19	−11·07	4·65	−6·17	3·68	−3·56	2·46	−2·31	1·82	−1·09
0·92	−7·61	−10·7	0·661	−11·17	3·06	−7·35	2·59	−4·72	2·03	−3·81
0·90	−10·1	−2·73	−6·51	−10·81	−0·64	−10·06	1·22	−7·16	1·24	−5·84
0·88	−6·38	0·668	−10·04	−5·39	−3·47	−9·73	−1·52	−8·62	−0·601	−7·35
0·86	−4·48	0·221	−8·64	−0·755	−8·82	−6·54	−4·72	−8·34	−3·11	−7·79
0·84	−4·00	0·0023	−5·96	0·608	−9·01	−2·64	−7·19	−6·44	−5·51	−6·88
0·82	−3·66	0·0094	−4·53	0·287	−7·39	−0·138	−8·17	−3·78	−7·06	−5·04
0·80	−3·40	0·005	−4·08	0·030	−5·57	0·533	−7·71	−1·42	−7·48	−2·90
0·75	−2·90	0·0	−3·40	0·007	−3·89	0·039	−4·74	0·423	−5·48	0·176
0·70	−2·66	0·0	−2·93	0·0	−3·30	0·008	−3·64	0·0544	−3·97	0·223
0·65	−2·50	0·0	−2·70	0·0	−2·88	0·0	−3·18	0·0036	−3·29	0·0094
0·60	−2·39	0·0	−2·54	0·0	−2·67	0·0	−2·89	0·003	−3·02	0·0046
0·55	−2·30	0·0	−2·41	0·0	−2·51	0·0	−2·59	0·0	−2·61	0·0
0·50	−2·23	0·0	−2·32	0·0	−2·39	0·0	−2·45	0·0	−2·46	0·0
0·45	−2·18	0·0	−2·24	0·0	−2·30	0·0	−2·34	0·0	−2·35	0·0
0·40	−2·14	0·0	−2·19	0·0	−2·23	0·0	−2·26	0·0	−2·26	0·0
0·35	−2·10	0·0	−2·14	0·0	−2·17	0·0	−2·19	0·0	−2·19	0·0
0·30	−2·07	0·0	−2·10	0·0	−2·12	0·0	−2·13	0·0	−2·14	0·0
0·25	−2·05	0·0	−2·07	0·0	−2·08	0·0	−2·09	0·0	−2·09	0·0
0·20	−2·03	0·0	−2·04	0·0	−2·05	0·0	−2·06	0·0	−2·06	0·0
0·15	−2·02	0·0	−2·02	0·0	−2·03	0·0	−2·03	0·0	−2·03	0·0
0·10	−2·01	0 0	−2·01	0·0	−2·01	0·0	−2·01	0·0	−2·01	0·0
0·05	−2·00	0·0	−2·00	0·0	−2·00	0·0	−2·00	0·0	−2·00	0·0
0·00	−2·00	0·0	−2·00	0·0	−2·00	0·0	−2·00	0·0	−2·00	0·0

TABLE 2. Real and imaginary parts $R(\tilde{\phi})$, $I(\tilde{\phi})$, respectively, of the adjoint function $\tilde{\phi}$.

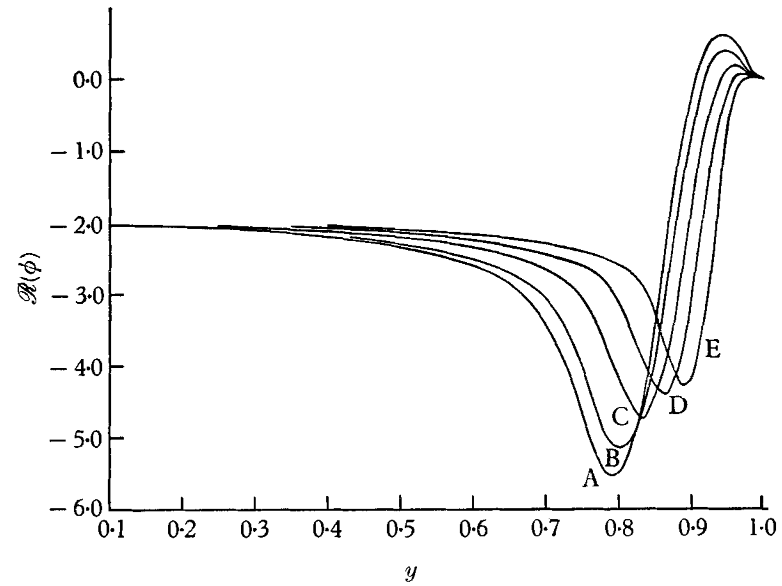

FIGURE 1. The real part $\mathscr{R}(\tilde{\phi})$ of $\tilde{\phi}$.

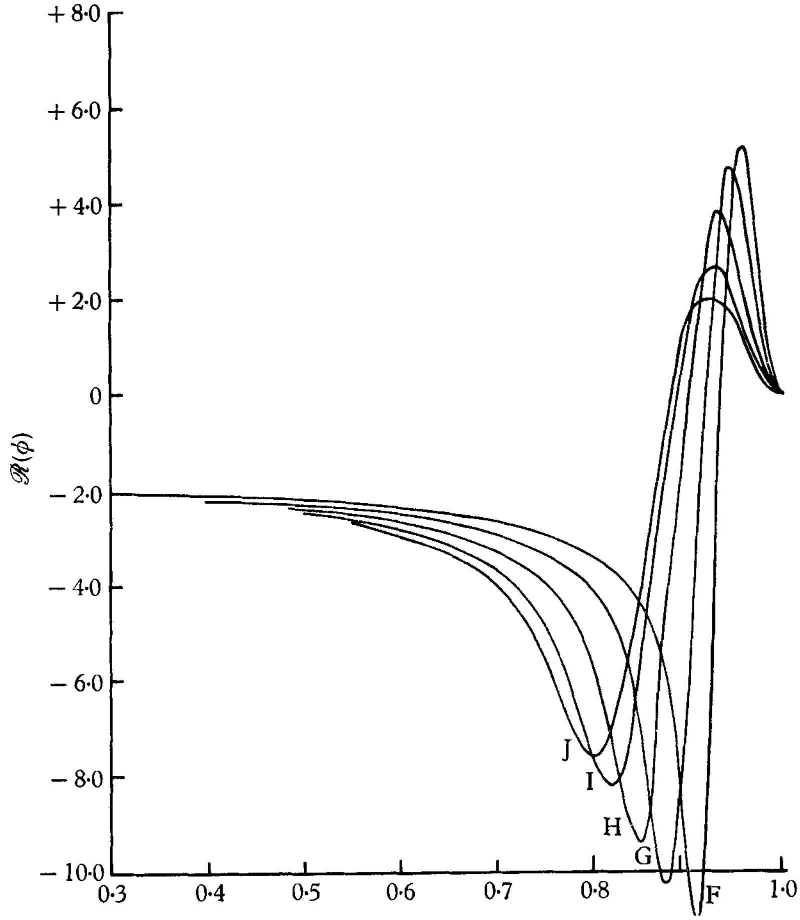

FIGURE 2. The real part $\mathscr{R}(\tilde{\phi})$ of $\tilde{\phi}$, continued.

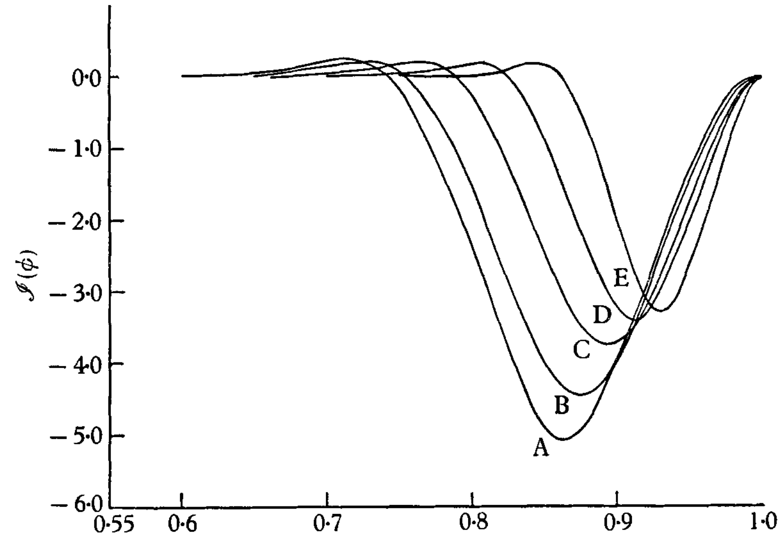

FIGURE 3. The imaginary part $\mathscr{I}(\tilde{\phi})$ of $\tilde{\phi}$.

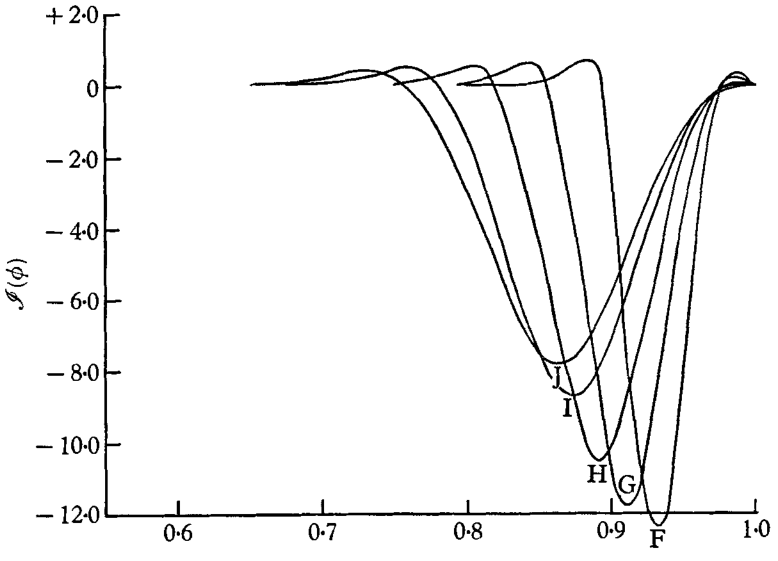

FIGURE 4. The imaginary part $\mathscr{I}(\tilde{\phi})$ of $\tilde{\phi}$, continued.

5. Calculation of the perturbed neutral stability curve

Having found the functions $\tilde{v}_1, \tilde{v}_3$, and knowing $\phi = v_1 + \delta v_3$, it is easy to calculate $(D^2 - \alpha^2)\phi = \tilde{v}_1 + \delta\tilde{v}_3$, and we are then in a position to work out the integrals in equation (10), once a value for SR has been assigned. The computation of these integrals for each of the points A, ..., J, for a series of values of SR from 0 to 1000 was performed on the Mercury computer of the University of London Computer Unit, using Simpson's rule on the integrand values for $y = 0$ (0·05) 0·8 (0·02) 1·0.

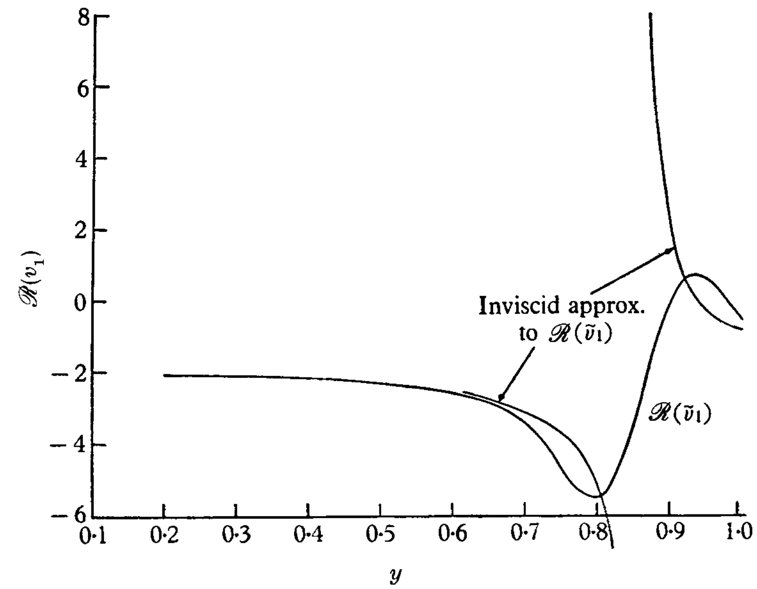

FIGURE 5. Case A. The function $\mathscr{R}(\tilde{v}_1)$ and the inviscid approximation to $\mathscr{R}(\tilde{v}_1)$.

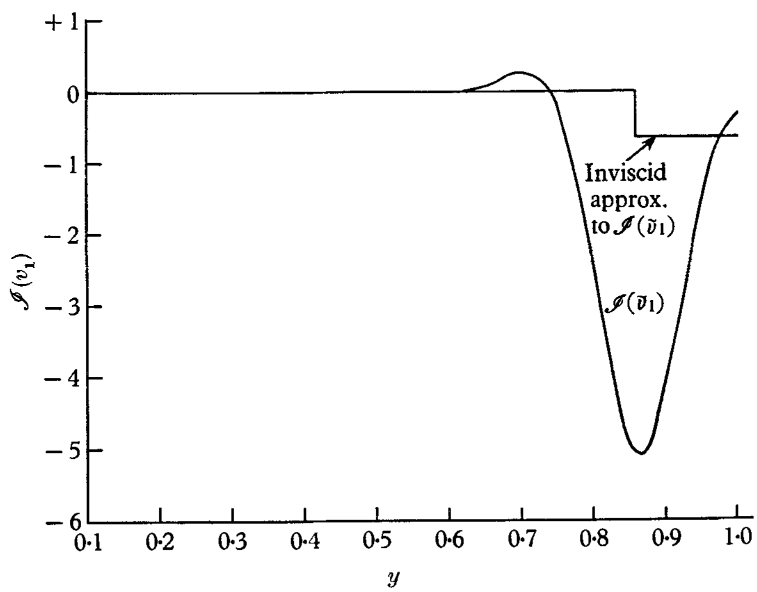

FIGURE 6. Case A. The function $\mathscr{I}(\tilde{v}_1)$ and the inviscid approximation to $\mathscr{I}(\tilde{v}_1)$.

It then remains to select the direction in which the perturbation is made by choosing arbitrarily a linear relation between λ, μ, ν, which together with the two real equations of (10) enable us to solve for λ, μ and ν.

Some remarks on the mathematics of the perturbation procedure are relevant here. If we consider the perturbations from the point P on the clean gas neutral stability curve in the (α, R)-plane we have seen that a straight line of perturbation points is obtained. This line of points is parallel to the tangent to the unperturbed curve at P, and the perturbed curve will be the envelope of the family of lines obtained by varying P. Each straight line will give a good approximation to a small section of the perturbed curve near its point of contact with this curve, but not at points a long way from the point of contact. This must be borne in mind when interpreting perturbation coefficients if a direction of perturbation is

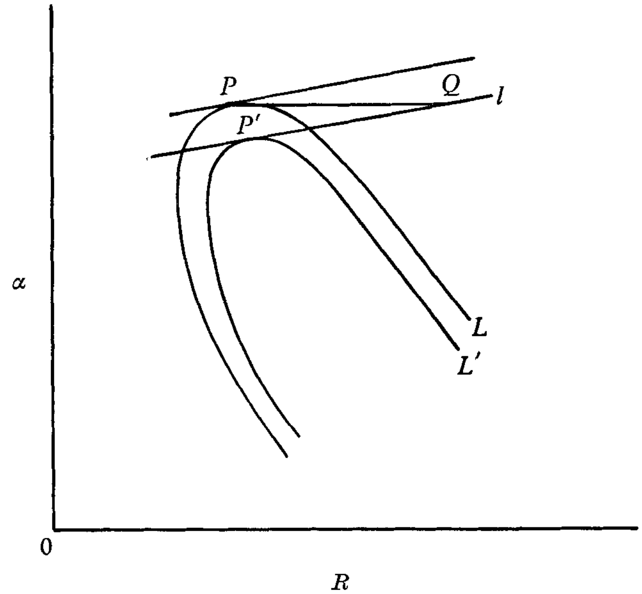

FIGURE 7. Illustration of a difficulty in the perturbation procedure.

chosen uniformly for all points of the original curve. A difficulty which arises may be illustrated by choosing, for example, perturbations at constant α for which $\lambda = 0$. Let L denote the unperturbed curve, L' the perturbed curve in figure 7. P is a point on L at which the tangent is nearly parallel to the R-axis, and l is the perturbation line which touches L' at P'. In such a case the perturbation at constant α will give the point Q, which is a long way from P' and L'. Clearly, for a given direction of perturbation in the (α, R)-plane points at which the tangent to the original curve is in the direction of perturbation will be singular points for that perturbation. Nevertheless, the simplest way to get the curve L' is to specify uniformly a direction of perturbation and to use a second form of perturbation to fill in the gaps near singular points of the first. The results given here rely mainly on the perturbation at constant α. The reason for using this is that the curve is of most interest near the points where R is least, and this part of the curve is well approximated by a perturbation at constant α. Perturbations at constant c were used to help bridge the gap near the singular point for constant α. The identification of spurious points of the perturbation depends on the value of f, and the author has studied the perturbation for $f = 0{\cdot}05$.

Interpretation of the perturbation in this case is given in the graphs of figures 8 and 9, which show the curves of neutral stability for a sequence of values of SR. For comparison the clean gas curve $f = 0$ is given in each case by the broken line;

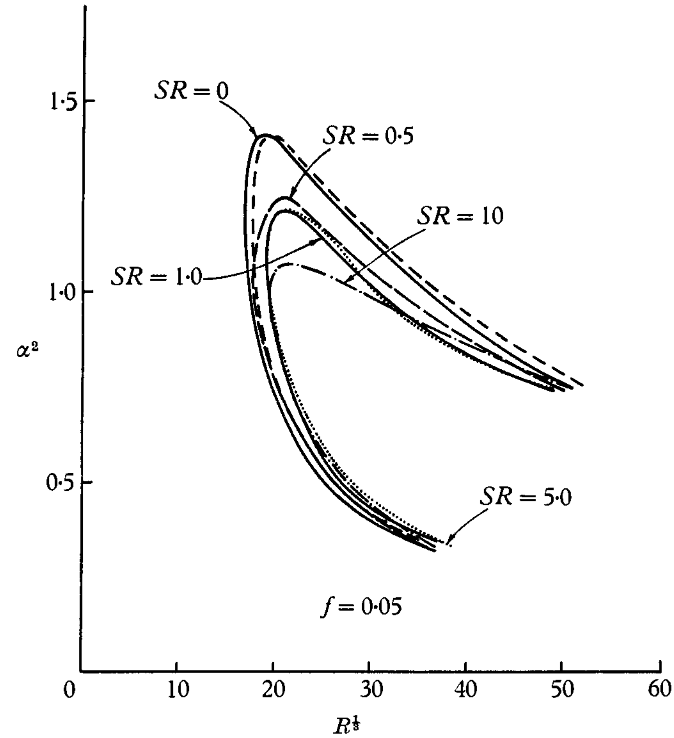

FIGURE 8. Curves of neutral stability for $f = 0{\cdot}05$.

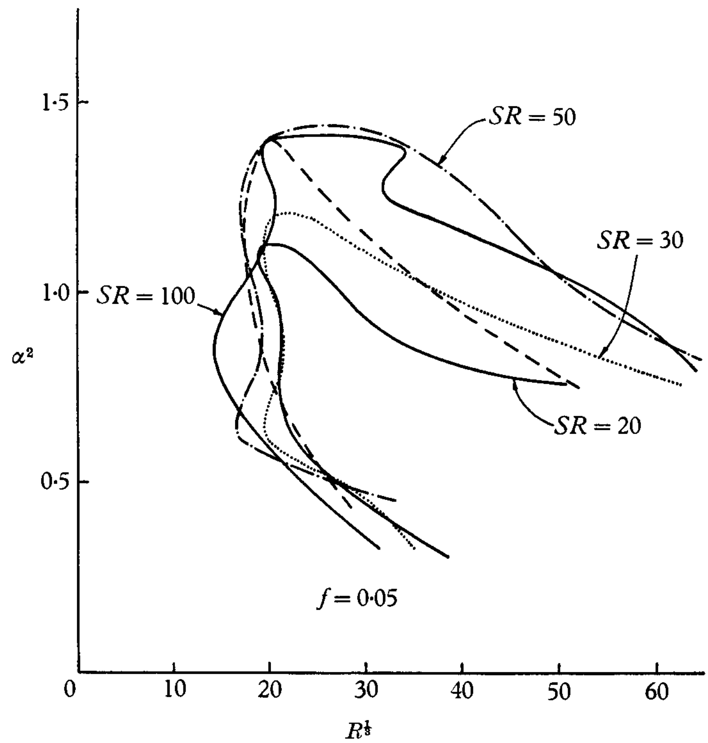

FIGURE 9. Curves of neutral stability for $f = 0{\cdot}05$.

the curve for $SR = 500$ cannot be distinguished from $f = 0$, on the scales used. Following the usual practice we have plotted α^2 against $R^{\frac{1}{3}}$.

At $SR = 0$ figure 8 illustrates the reduction of R in the ratio $1/(1+f)$ as deduced by Saffman. As SR increases a feature of the curves is the reduction in the maximum wave-number for instability representing an increase in the minimum size of unstable wave cells. This effect is at its strongest for $SR \sim 10$ when the maximum value of α^2 is reduced to about 1·05. This increase in cell size may be explained in general terms by the increasing time delay of the dust in following the gas fluctuation velocity as SR increases since the dust will be less able to follow small scale motions in which the changes in direction of the fluctuation velocity of the gas are more pronounced, and failure of the dust to follow smaller scale fluctuations will stabilize the disturbance. As SR increases to higher values the range of α^2 begins to increase again towards the limiting case in which $f/SR \to 0$, which represents dust of vanishing size or vanishing number density.

The results also show that the critical value of $R^{\frac{1}{3}}$ remains approximately 19 in the range $1 < SR < 30$. At higher values of SR a distinct instability arises at low wave-numbers, and at $SR = 100$, for example, $R_c^{\frac{1}{3}} \sim 14$ at $\alpha^2 \sim 0{\cdot}85$. Another feature of the behaviour for large SR which is not understood at present is the wide extension of the upper branch of the curves for $SR = 50, 100$, to take in much higher values of R into the unstable region. However, this aspect of the results must be viewed with some reserve, since, quite apart from errors introduced in the approximations, the perturbations are clearly too large at this value of f for the results to be reliable.

Perturbation curves were also studied for $f = 0{\cdot}1$, but in general the perturbations are then too large and erratic for this method to be of use. However, the results obtained did suggest that for some values of SR in this case, the curve of neutral stability is in two parts, one of which is a closed loop giving an island of instability, such as those obtained recently by Miles (1960). But the method used here is not powerful enough to deduce such complicated perturbations with any certainty and accuracy.

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